UDC 539.3

DYNAMIC CONTACT PROBLEM FOR A VISCO-ELASTIC HALF-PLANE

PMM Vol. 41, № 5, 1977, pp. 926-934

V. P. TEN
(Kemerovo)

(Received December 15, 1976)

Steady oscillations of a rigid stamp at the boundary of a visco-elastic half-plane are considered. No tractions exist outside the region of contact, and the absence of friction, Coulomb friction or coupling of the stamp to the half-plane all prevail within the region. A system of integral equations of the first kind obtained with the help of a Fourier transformation is reduced, by differentiating with respect to the coordinate and separating the kernal singularities, to a system of singular integral equations. At zero oscillation frequency the latter system coincides with the equations of the analogous static problem of the theory of elasticity. The exact solutions of the static problem are utilized for regularization of the system according to Carleman-Vekua when the oscillation frequency is not zero.

The low frequency asymptotic of the system kernals is investigated using the contour integration, and the asymptotic properties of the Laplace transformation. The solution of the system is constructed in first approximation for low frequency oscillations. Oscillations of a stamp on an elastic half-plane were studied in [1, 2], while the results of the problem with coupling were presented at the All- Union Winter School (*).

1. Basic equations. We shall assume that the visco-elastic medium fills the half-plane $y \leqslant 0$. We derive the basic relations under the condition that the displacements within the area of contact $|x| < a$ are given, and there are no stresses outside this area $\sigma_y = \tau_{xy} = 0$.

The complex amplitudes of the displacements of the viscoelastic half-plane satisfy the following system of equations:

$$(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\Delta u = -\rho\omega^2 u \quad \left(\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{1.1}$$

$$(\lambda + \mu)\frac{\partial\theta}{\partial y} + \mu\Delta v = -\rho\omega^2 v$$

The amplitudes of the displacements are connected with the stress amplitudes by the relations

$$\sigma_y = \lambda\theta + 2\mu\frac{\partial v}{\partial y}, \quad \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{1.2}$$

$$\lambda = \lambda_0\left[1 - \int_0^\infty \Lambda(x)\, e^{-i\omega x}dx\right], \quad \mu = \mu_0\left[1 - \int_0^\infty M(x)\, e^{-i\omega x}dx\right]$$

(*)Zlatina I.N. and Zil'bergleit A.S. Use of dual integral equations in the dynamic contact problem of oscillations of a rigid stamp on an elastic half-space. In " Contact Problems of the Mechanics of Deformable Solids ". Erevan, 1976.

Here λ_0 and μ_0 are the instantaneous moduli of elasticity of the medium, Λ and M are the completeness kernals, and ω is the frequency of the steady oscillations. We assume that in the limit case of $\omega = 0$ the complex moduli λ and μ are not zero.

We attach to (1.1) the following boundary conditions:

$$u = f(x), \quad v = g(x) \quad \text{for} \quad y = 0, \ |x| < a \tag{1.3}$$
$$\sigma_y = \tau_{xy} = 0 \quad \text{for} \quad y = 0, \ |x| \geqslant a$$

and introduce the dimensionless coordinates, displacements and stresses

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{a}, \quad U = \frac{u}{a}, \quad V = \frac{v}{a}, \quad s_\eta = \frac{1}{\mu}\sigma_y, \ t_{\xi\eta} = \frac{1}{\mu}\tau_{xy}$$

We denote the limiting values of s_η and $t_{\xi\eta}$ within the area of contact by $p(\xi)$ and $q(\xi)$ respectively.

Applying to (1.1) - (1.3) the generalized Fourier transformation [3] in ξ, we obtain a system of ordinary differential equations and the corresponding initial conditions at $\eta = 0$. Having solved this system, we perform the inverse transformation and differentiate the resulting expressions with respect to ξ, we obtain in the area of contact the following system of integral equations of the first kind:

$$\beta V'(\xi, 0) = \frac{i}{4\pi}\int_{-1}^{1} G_{11}(x-\xi)\,p(x)\,dx - \frac{n^2}{4\pi}\int_{-1}^{1} G_{12}(x-\xi)\,q(x)\,dx \tag{1.4}$$

$$\beta U'(\xi, 0) = -\frac{n^2}{4\pi}\int_{-1}^{1} G_{21}(x-\xi)\,p(x)\,dx - \frac{i}{4\pi}\int_{-1}^{1} G_{22}(x-\xi)\,q(x)\,dx$$

$$\left(\beta = \frac{\lambda+\mu}{\lambda+2\mu}, \quad n^2 = \frac{\mu}{\lambda+2\mu}, \quad G_{ij}(z) = \int_{-\infty}^{\infty} g_{ij} e^{ikz\zeta} k\,d\zeta, \right.$$
$$\left. k = a\omega\left|\frac{\rho}{\mu}\right|^{1/2}\right)$$

where the functions $g_{ij}(\zeta, k_1, k_2)$ have the form

$$g_{11} = \frac{(1-n^2)\zeta r_1}{2R(\zeta, k_1, k_2)}, \quad g_{22} = \frac{(1-n^2)\zeta r_2}{2R(\zeta, k_1, k_2)}$$

$$g_{12} = g_{21} = \frac{\lambda+\mu}{\mu}\,\frac{(\varkappa - r_1 r_2)\zeta^2}{R(\zeta, k_1, k_2)}$$

$$r_1 = \sqrt{\zeta^2 - k_1^2}, \quad r_2 = \sqrt{\zeta^2 - k_2^2}, \quad \varkappa = \zeta^2 - {}^1/_2 k_2^2, \quad R = r_1 r_2 \zeta^2 - \varkappa^2$$

$$\left(k_1^2 = \frac{|\mu|}{\lambda+2\mu}, \quad k_2^2 = \frac{|\mu|}{\mu}, \quad -\frac{\pi}{4} < \arg k_{1,2} \leqslant 0\right)$$

The stresses will vanish at infinity only if the roots can be expanded on the real axis (this automatically ensures that the principle of irradiation holds in the elastic solution)

$$r_1 = |\zeta|\left(1 - \frac{k_1^2}{2\zeta^2} - \frac{k_1^4}{8\zeta^4} - \ldots\right), \quad r_2 = |\zeta|\left(1 - \frac{k_2^2}{2\zeta^2} - \frac{k_2^4}{8\zeta^4} - \ldots\right)$$

This yields the following expansions for the functions g_{ij} when $|\zeta| \gg 1$:

$$g_{11} = \operatorname{sign}\zeta\left[1 + \frac{3-4n^2+3n^4}{4(1-n^2)}\frac{k_2^2}{\zeta^2} + \ldots\right] \tag{1.5}$$

$$g_{22} = \operatorname{sign}\zeta\left[1 + \frac{1-n^2+n^4}{4(1-n^2)}\frac{k_2^2}{\zeta^2} + \ldots\right]$$

$$g_{12} = g_{21} = 1 + \frac{1+n^4}{4n^2(1-n^2)}\frac{k_2^2}{\zeta^2} + \ldots$$

Using the above relations and the formulas [3]

$$\int_{-\infty}^{\infty} 1e^{it(x-\xi)}dt = 2\pi\delta(x-\xi), \qquad \int_{-\infty}^{\infty} \operatorname{sign} t e^{it(x-\xi)}dt = \frac{2i}{x-\xi}$$

we can reduce the kemals of the system (1.4) to the form

$$G_{nn} = \frac{2i}{x-\xi} + h_{nn}(x-\xi), \quad G_{12} = G_{21} = 2\pi\delta(x-\xi) + h_{12}(x-\xi)$$

$$h_{nn}(z) = \int_{-\infty}^{\infty} (g_{nn} - \operatorname{sign}\zeta)e^{ikz\zeta}kd\zeta \quad (n = 1, 2)$$

$$h_{12}(z) \int_{-\infty}^{\infty} (g_{12} - 1)e^{ikz\zeta}kd\zeta$$

Since the functions g_{11}' and g_{22} are odd and g_{12} is given in ζ, the integ - ration can only be performed in the interval $(0, \infty)$, and this enables us to consider a single branch of the roots r_1 and r_2. We omit the derivation, and quote the system (1.4) in its final form

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{p(x)}{x-\xi}dx + \frac{1}{2\pi}\int_{-1}^{1}h_{11}^*p(x)\,dx + \frac{n^2}{2}q(\xi) + \frac{n^2}{2\pi}\int_{-1}^{1}h_{12}^*q(x)\,dx = \psi \tag{1.6}$$

$$-\frac{1}{2\pi}\int_{-1}^{1}\frac{q(x)}{x-\xi}dx - \frac{1}{2\pi}\int_{-1}^{1}h_{22}^*\,q(x)\,dx + \frac{n^2}{2}p(\xi) +$$

$$\frac{n^2}{2\pi}\int_{-1}^{1}h_{21}^*\,p(x)\,dx = \varphi$$

$$\left(\psi = -\frac{\beta}{a}g_{\xi}', \quad \varphi = -\frac{\beta}{a}f_{\xi}'\right)$$

Its kemals

$$h_{nn}^* = \int_0^{\infty}(g_{nn} - 1)\sin[k(x-\xi)\zeta]\,kd\zeta \quad (n = 1, 2) \tag{1.7}$$

$$h_{12}^* = h_{21}^* = \int_0^{\infty}(g_{12} - 1)\cos[k(x-\xi)\zeta]\,kd\zeta$$

are continuous by virtue of the estimates (1.5)

The system (1.6) can be easily regularized according to Carleman - Vekua, since at $\omega = 0$ (1. 6) transforms into the usual static equations of the theory of elasticity. The final result of the regularization depends on the type of the conditions prevailing under the stamp.

2. **Estimation of the kernals of (1.6).** Let us consider, for definiteness, the kernal $h_{11}^*(z)$. The concept of estimating a kernal consists of the passage, in the course of integration, from the real axis to the imaginary axis.

Let us separate the real part of h_{11}^* from the imaginary part. We define the branches of the roots r_1 and r_2 in the right half-plane of the complex variable ζ by the expansions

$$r_1 = \zeta - \frac{k_1^2}{2\zeta} - \dots,$$

$$r_2 = \zeta - \frac{k_2^2}{2\zeta} - \dots$$

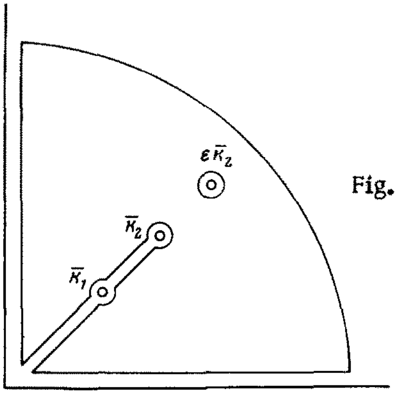

Fig. 1

Similarly we define the branches of the conjugate $\bar{k}_1$ and $\bar{k}_2$ by

$$r_1^* = \sqrt{\zeta^2 - \bar{k}_1^2} = \zeta - \frac{\bar{k}_1^2}{2\zeta} - \dots,$$

$$r_2^* = \sqrt{\zeta^2 - \bar{k}_2^2} = \zeta - \frac{\bar{k}_2^2}{2\zeta} - \dots$$

On the real axis, r_n and r_n^* are complex conjugate. This enables us to separate the real from the imaginary parts of h_{11}^* in the form

$$\operatorname{Re} h_{11}^* = -\frac{k}{2} \operatorname{sign} z \operatorname{Im} \int_0^\infty [g_{11}(\zeta, k_1, k_2) + g_{11}(\zeta, \bar{k}_1, \bar{k}_2) - 2]\, e^{ik|z|\zeta} d\zeta$$

$$\operatorname{Im} h_{11}^* = -\frac{k}{2} \operatorname{sign} z \operatorname{Re} \int_0^\infty [g_{11}(\zeta, k_1, k_2) - g_{11}(\zeta, \bar{k}_1, \bar{k}_2)]\, e^{ik|z|\zeta} d\zeta$$

In the first quadrant of the ζ-plane the integrand functions have branch points $\bar{k}_1$ and $\bar{k}_2$ and a pole $\varepsilon\bar{k}_2$ at the zero of the Rayleigh function $R(\zeta, \bar{k}_1, \bar{k}_2)$. They decrease at infinity as ζ^{-2} by virtue of the estimates (1.5).

In order to simplify the calculations, we assume that $\Lambda \equiv \mathrm{M}$. Then the cuts around $\bar{k}_1$ and $\bar{k}_2$ will merge and n will become real. Integrating along the contour shown in the Fig. 1, we obtain, after combining the real and imaginary parts of h_{11}^*.

$$h_{11}^* = k \operatorname{sign} z \left\{ \frac{1-n^2}{2} \left[R_{11} k_2 e^{iw\varepsilon} + \int_0^n P_{11} e^{iwt} dt + \int_n^1 S_{11} e^{iwt} dt \right] + \int_0^\infty [g_{11}(it, k_1, k_2) - 1]\, e^{-k|z|t} dt \right\} \qquad (2.1)$$

$$R_{11} = 2\pi(\varepsilon^2 - n^2)\sqrt{\varepsilon^2 - 1}\, r, \quad P_{11} = t\sqrt{n^2 - t^2}\, l,$$

$$S_{11} = (t^2 - n^2)\sqrt{1 - t^2}\, m$$

$$r^{-1} = 4\varepsilon^4 - 3(1 + n^2)\varepsilon^2 + 2n^2 - 4(\varepsilon^2 - {}^1/_2)\sqrt{\varepsilon^2 - 1}\sqrt{\varepsilon^2 - n^2}, \quad w = k|z|k_2$$

$$l^{-1} = ({}^1/_2 - t^2)^2 + t^2\sqrt{1 - t^2}\sqrt{n^2 - t^2},$$

$$m^{-1} = t^4(t^2 - n^2)(1 - t^2) + (t^2 - {}^1/_2)^4$$

Similarly we obtain

$$h_{22}^* = k \operatorname{sign} z \left\{ \frac{1-n^2}{2} \left[R_{22} k_2 e^{i\omega\varepsilon} + \int_0^n P_{22} e^{i\omega t} dt + \int_n^1 S_{22} e^{i\omega t} dt \right] + \right. \tag{2.2}$$

$$\left. \int_0^\infty [g_{22}(it, k_1, k_2) - 1] e^{-k|z|t} dt \right\}$$

$$R_{22} = 2\pi(\varepsilon^2 - 1)\sqrt{\varepsilon^2 - n^2}\, r, \quad P_{22} = t\sqrt{1-t^2}\, l,$$

$$S_{22} = t(1-t^2)\sqrt{t^2 - n^2}\, m$$

Unlike h_{11}^* and h_{22}^*, the kernal h_{12}^* has a single residue at the point εk_2 and an integral along the cut edges $(\bar{k}_1\ \bar{k}_2)$

$$h_{12}^* = -\frac{1-n^2}{2n^2} kik_2 \left[R_{12} e^{i\omega\varepsilon} + \int_n^1 S_{12} e^{i\omega t} dt \right] \tag{2.3}$$

$$R_{12} = 2\pi\left[\sqrt{\varepsilon^2 - 1}\sqrt{\varepsilon^2 - n^2}(\varepsilon^2 - {}^1/_2) + (1+n^2)\varepsilon^2 - n^2 - \varepsilon^4\right] r$$

$$S_{12} = \sqrt{1-t^2}\sqrt{t^2 - n^2}(t^2 - {}^1/_2)\, t^2 m$$

when $k \ll 1$ the estimate for h_{12}^* can be obtained by expanding the exponential part in (2.3) into a series. In the first approximation we have

$$h_{12}^* = -i\frac{1-n^2}{2n^2}\left(R_{12} + \int_n^1 S_{12} dt\right) k_2 k = -ih \tag{2.4}$$

The expressions within the curly brackets in (2.1) and (2.2) vanish when $k = 0$. Using this fact, the estimates (1.5) and the asymptotic properties of the Laplace transfromation [4], we obtain

$$h_{11}^* = -\frac{3 - 4n^2 + 3n^4}{4(1-n^2)} k_2{}^2 z k^2 \ln\frac{1}{k} = -bz \tag{2.5}$$

$$h_{22}^* = -\frac{1 - n^2 + n^4}{4(1-n^2)} k_2{}^2 z k^2 \ln\frac{1}{k} = -cz$$

3. Contact problems for a plane stamp. Let us investigate the low frequency oscillations of a stamp acted upon by the forces and moment with the amplitudes:

$$P_0 = \int_{-1}^1 p(\xi)\, d\xi, \quad Q_0 = \int_{-1}^1 q(\xi)\, d\xi, \quad M_0 = -\int_{-1}^1 \xi p(\xi)\, d\xi$$

We denote the amplitude of the angle of rotation of the stamp by α, so that

$$\psi = -\frac{\lambda+\mu}{\lambda+2\mu}\alpha, \quad \varphi = 0$$

We assume that frictional forces are absent from the area of contact. In this case $q(x) \equiv 0$ and the system(1.6) degenerates into a single equation

$$\frac{1}{2\pi}\int_{-1}^1 \frac{p(x)}{x-\xi} dx = \frac{b}{2\pi}\int_{-1}^1 (x-\xi)\, p(x)\, dx - \frac{\lambda+\mu}{\lambda+2\mu}\alpha \tag{3.1}$$

When $k = 0$, (3.1) becomes an equation of statics. Using the static solution [5] we regularize (3.1) according to Carleman - Vekua [6]. We omit the derivation to give the final result

$$p(\xi) = \frac{P_0}{\pi\sqrt{1-\xi^2}} - \frac{2(\lambda+\mu)\alpha\xi}{(\lambda+2\mu)\sqrt{1-\xi^2}} + \frac{b}{\pi}\int_{-1}^{1}\frac{x\xi-\xi^2+{}^1/_2}{\sqrt{1-\xi_2}}\,p(x)\,dx \qquad (3.2)$$

In the first approximation the solution (3.2) has the form

$$p(\xi) = \frac{P_0}{\pi\sqrt{1-\xi^2}}\left[1+\frac{b}{2}(1-2\xi^2)\right] - \frac{2(\lambda+\mu)\alpha\xi}{(\lambda+2\mu)\sqrt{1-\xi^2}}\left(1+\frac{b}{2}\right)$$

$$\left(\frac{b}{2} = \frac{3-4n^2+3n^4}{8(1-n^2)}k_2^2k^2\ln\frac{1}{k}\right)$$

The solution obtained enables us to find the relation between the amplitudes of the force moment, and the angle of rotation of the stamp. Integrating we obtain

$$M_0 = \pi\alpha\frac{\lambda+\mu}{\lambda+2\mu}\left(1+\frac{b}{2}\right)$$

Let the frictional forces obeying the Coulomb law $(0 < \wedge)\ (x)d\wedge = (x)\ b$, act in the area of contact. The system (1.6) reduces to the single equation

$$\frac{\nu n^2}{2}p(\xi) + \frac{1}{2\pi}\int_{-1}^{1}\frac{p(x)}{x-\xi}\,dx = \psi + \frac{1}{2\pi}\int_{-1}^{1}[b(x-\xi)+i\nu n^2h]\,p(x)\,dx \qquad (3.3)$$

Let us denote the right hand side of (3.3) by $F(\xi)$. Introducing the analytic function

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{p(x)}{x-z}\,dx$$

we obtain. in place of (3.3), the problem of conjugation on the interval $(-1,1)$

$$\Phi^+ = -\frac{i-\nu n^2}{i+\nu n^2}\Phi^- + \frac{2}{i+\nu n^2}F$$

the solution of which is

$$\Phi(z) = -\frac{P_0}{2\pi i}X(z) + \frac{2}{i+\nu n^2}\frac{X(z)}{2\pi i}\int_{-1}^{1}\frac{F(\xi)\,d\xi}{X^+(\xi)(\xi-z)}$$

where X is a branch of the function

$$X(z) = (z+1)^{-1/2-\theta}(z-1)^{-1/2+\theta} \quad \left(e^{2\pi i\theta} = \frac{i-\nu n^2}{i+\nu n^2},\quad 0 \leqslant \theta < \frac{1}{2}\right)$$

defined by the expansion $X = 1/z + \ldots$ at infinity.

Computing the integrals and jumps of the functions, we arrive at the equation

$$p(\xi) = \left(\frac{1-\xi}{1+\xi}\right)^{\theta}\frac{\cos\pi\theta}{\sqrt{1-\xi^2}}\left[\frac{P_0}{\pi} + 2\psi(\xi+2\theta) + 2\int_{-1}^{1}K(x,\xi)\,p(x)\,dx\right] \qquad (3.4)$$

$$K(x,\xi) = (bx + i\nu n^2h)(\xi+2\theta) - (\xi^2 + 2\theta\xi - {}^1/_2 + 2\theta^2)b$$

The solution (3.4) has the following form in first approximation:

$$p(\xi) = \left(\frac{1-\xi}{1+\xi}\right)^{\theta}\frac{\cos\pi\theta}{\sqrt{1-\xi^2}}\left\{\frac{P_0}{\pi}\left[1 + i\nu n^2h\left(\theta+\frac{1}{2}\xi\right) + \right.\right.$$

$$+\frac{b}{2}(1-2\xi^2-8\xi\theta-12\theta^2)\Big]-\frac{2(\lambda+\mu)}{\lambda+2\mu}\alpha(\xi+2\theta)\times$$
$$\Big[1+\frac{b}{2}(1-4\theta^2)\Big]\Big]$$

and at $k=0$ it becomes identical with the static solution [5].

Calculating the moment of forces, we find the relation connecting M_0, P_0 and α

$$M_0=P_0\Big[2\theta-ivn^2h\Big(\frac{1}{4}-\theta^2\Big)+\theta b\,\frac{9-32\theta^2}{6}\Big]+\frac{\lambda+\mu}{\lambda+2\mu}\pi\alpha(1-4\theta^2)\Big[1+\frac{b}{2}(1-4\theta^2)\Big]$$

The solutions of the previous problems have a physical meaning only in the case when static solutions which ensure that the pressure under the stamp is not negative, are imposed upon them. Let us now assume that the stamp is coupled to the half-plane. Let us perform the change of variables

$$p=\varphi_1+\varphi_2,\quad q=-i(\varphi_1-\varphi_2) \tag{3.5}$$

and pass, for convenience, from the system (1.6) to

$$n^2\varphi_1-\frac{1}{\pi i}\int_{-1}^{1}\frac{\varphi_1dx}{x-\xi}-\frac{1}{\pi i}\int_{-1}^{1}k_{11}\varphi_1dx-\frac{1}{\pi i}\int_{-1}^{1}k_{12}\varphi_2dx=i\psi \tag{3.6}$$

$$n^2\varphi_2+\frac{1}{\pi i}\int_{-1}^{1}\frac{\varphi_2dx}{x-\xi}+\frac{1}{\pi i}\int_{-1}^{1}k_{22}\varphi_2dx+\frac{1}{\pi i}\int_{-1}^{1}k_{21}\varphi_1dx=-i\psi$$

$$\Big(k_{11}=\frac{h_{11}^*+h_{22}^*}{2}-in^2h_{12}^*,\ k_{22}=\frac{h_{11}^*+h_{22}^*}{2}+in^2h_{12}^*$$

$$k_{12}=k_{21}=\frac{h_{11}^*-h_{22}^*}{2}\Big)$$

In accordance with the estimates (2.4) and (2.5), the kernals of the system (3.6) have the following form in first approximation:

$$k_{11}=-n^2h-\frac{b+c}{2}(x-\xi),\quad k_{22}=n^2h-\frac{b+c}{2}(x-\xi) \tag{3.7}$$

$$k_{12}=-\frac{b-c}{2}(x-\xi)$$

Let us regularize the system (3.6). Introducing the analytic finctions

$$\Phi_1=\frac{1}{2\pi i}\int_{-1}^{1}\frac{\varphi_1dx}{x-z},\quad \Phi_2=\frac{1}{2\pi i}\int_{-1}^{1}\frac{\varphi_2dx}{x-z}$$

we obtain, in place of (3.6), the problems of conjugation

$$\Phi_1^+=-\frac{1+n^2}{1-n^2}\Phi_1^--\frac{A_1}{1-n^2}$$

$$\Big(A_1=i\psi+\frac{1}{\pi i}\int_{-1}^{1}k_{11}\varphi_1dx+\frac{1}{i}\int_{-1}^{1}k_{12}\varphi_2dx\Big)$$

$$\Phi_2^+=-\frac{1-n^2}{1+n^2}\Phi_2^-+\frac{A_2}{1+n^2}$$

$$\left(A_2 = -i\psi - \frac{1}{\pi i}\int_{-1}^{1} k_{22}\varphi_2 dx - \frac{1}{\pi i}\int_{1}^{1} k_{21}\varphi_1 dx\right)$$

Their solutions have the form

$$\Phi_1(z) = c_1 X_1(z) - \frac{1}{1-n^2}\frac{X_1(z)}{2\pi i}\int_{-1}^{1}\frac{A_1 d\xi}{X_1^+(\xi)(\xi - z)} \qquad (3.8)$$

$$\Phi_2(z) = c_2 X_2(z) + \frac{1}{1+n^2}\frac{X_2(z)}{2\pi i}\int_{-1}^{1}\frac{A_2 d\xi}{X_2^+(\xi)(\xi - z)}$$

$$X_1 = (z+1)^{-1/2+i\gamma}(z-1)^{-1/2-i\gamma}, \quad X_2 = (z+1)^{-1/2-i\gamma}(z-1)^{-1/2+i\gamma}$$

$$\left(\gamma = \frac{1}{2\pi}\ln\frac{1+n^2}{1-n^2}, \quad X_1 = \frac{1}{z} + \ldots, \quad X_2 = \frac{1}{z} + \ldots\right)$$

$$c_1 = -\frac{1}{4\pi i}(P_0 + iQ_0), \quad c_2 = -\frac{1}{4\pi i}(P_0 - iQ_0)$$

Computing the quadratures and jumps in the values of the functions, we obtain

$$\varphi_1 = \left\{c_1 + \frac{i\psi}{2}(\xi - 2\gamma i) - \frac{1}{2\pi i}\int_{-1}^{1} k_{11}^*\varphi_1 dx - \frac{1}{2\pi i}\int_{-1}^{1} k_{12}^*\varphi_2 dx\right\}[X_1] \qquad (3.9)$$

$$\varphi_2 = \left\{c_2 + \frac{i\psi}{2}(\xi + 2\gamma i) - \frac{1}{2\pi i}\int_{-1}^{1} k_{22}^*\varphi_2 dx - \frac{1}{2\pi i}\int_{-1}^{1} k_{21}^*\varphi_1 dx\right\}[X_2]$$

$$k_{11}^* = \left(n^2 h + \frac{b+c}{2}x\right)(\xi - 2\gamma i) - \frac{b+c}{2}\left(\xi^2 - 2\gamma i\xi - \frac{1}{2} - 2\gamma^2\right)$$

$$k_{12}^* = \frac{b-c}{2}\left[(\xi - 2\gamma i)x - \xi^2 + 2\gamma i\xi + \frac{1}{2} + 2\gamma^2\right]$$

$$k_{22}^* = \left(-n^2 h + \frac{b+c}{2}x\right)(\xi + 2\gamma i) - \frac{b+c}{2}\left(\xi^2 + 2\gamma i\xi - \frac{1}{2} - 2\gamma^2\right)$$

$$k_{21}^* = \frac{b-c}{2}\left[(\xi + 2\gamma i)x - \xi^2 - 2\gamma i\xi + \frac{1}{2} + 2\gamma^2\right]$$

$$[X_1] = \frac{B}{i\sqrt{1-\xi^2}}e^{iA}, \quad [X_2] = \frac{B}{i\sqrt{1-\xi^2}}e^{-iA}$$

$$\left(B = \frac{2}{\sqrt{1-n^4}}, \quad A = \gamma\ln\frac{1+\xi}{1-\xi}\right)$$

In first approximation the solution (3.9) has the form

$$\varphi_1 = \left[c_1 a_1(\xi) + c_2 d(\xi) + \frac{i\psi}{2}(\xi - 2\gamma i)e\right][X_1]$$

$$\varphi_2 = \left[c_2 a_2(\xi) + c_1 d(\xi) + \frac{i\psi}{2}(\xi + 2\gamma i)e\right][X_2]$$

$$a_k(\xi) = 1 - 2n^2 h\gamma i + \frac{b+c}{2}\left(\frac{1}{2} - \xi^2 + 6\gamma^2\right) - (-1)^k[n^2 h + 2(b+c)\gamma i]\xi$$

$$d(\xi) = \frac{b-c}{2}\left(\frac{1}{2} - \xi^2 - 2\gamma^2\right), \quad e = 1 + b\left(\frac{1}{2} + 2\gamma^2\right) \qquad (k = 1,2)$$

Using (3.5) and (3.8), let us now return to p and q. Let the stamp be acted upon by a single vertical force of amplitude P_0. Then

$$p(\xi) = P_0^* [f_n(k) \cos A + f_{ns}(k) \xi \sin A]$$
$$q(\xi) = P_0^* [f_n(k) \sin A - f_{ns}(k) \xi \cos A], \quad P_0^* = \frac{BP_0}{2\pi\sqrt{1-\xi^2}}$$

If the stamp is acted upon by a single horizontal force of amplitude Q_0 then we have

$$p(\xi) = -Q_0^* [f_s(k) \sin A - f_{ns}(k) \xi \cos A]$$
$$q(\xi) = Q_0^* [f_s(k) \cos A + f_{ns}(k) \xi \sin A], \quad Q_0^* = \frac{BQ_0}{2\pi\sqrt{1-\xi^2}}$$

where

$$f_n(k) = 1 - 2n^2 h\gamma i + b(^1/_2 - \xi^2 + 2\gamma^2) + 4c\gamma^2$$
$$f_s(k) = 1 - 2n^2 h\gamma i + c(^1/_2 - \xi^2 + 2\gamma^2) + 4b\gamma^2$$
$$f_{ns}(k) = n^2 hi - 2(b + c)\gamma$$

Finally, if the stamp oscillates with the amplitudes α of the angle of rotation, then

$$p(\xi) = -\alpha^* (\xi \cos A + 2\gamma \sin A), \quad q(\xi) = -\alpha^* (\xi \sin A - 2\gamma \cos A)$$
$$\alpha^* = \frac{\lambda+\mu}{\lambda+2\mu} \frac{\alpha B}{\sqrt{1-\xi^2}} \left[1 + b\left(\frac{1}{2} + 2\gamma^2\right)\right]$$

Multiplying p by ξ and integrating, we obtain the relation connecting M_0 and α

$$M_0 = \frac{\lambda+\mu}{\lambda+2\mu} \pi\alpha \left[1 + b\left(\frac{1}{2} + 2\gamma^2\right)\right](1 + 4\gamma^2)$$

In conclusion, the author thanks L. A. Galin for the interest shown.

REFERENCES

1. Borodachev, N.M. Determination of stresses under an oscillating foundation. Osnovaniia, fundamenty i mekhanika gruntov. No. 3, 1962.

2. Buriak, V. G. Dynamic contact problem for an elastic half-plane. Izv. Akad. Nauk. SSSR, MTT, No. 6, 1972.

3. Vladimirov, V. S. Equations of Mathematical Physics. Moscow, "Nauka" 1976.

4. Dötsch, G., Handbuch der Laplace — Transformation, bd 3. Anw. Laplace —Transformation. Basel—Stuttgart, Birckhäuservlag, 1956.

5. Muskhelishvili, N. I., (English translation) Some Basic Problems of the Mathematical Theory of Elasticity. Groningen, Noordhoff, 1953.

6. Muskhelishvili, N. I., Singular Integral Equations, Moscow, "Nauka" 1968.

Translated by L. K.